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Bound states of *n*-dimensional harmonic oscillator decorated with Dirac delta functions

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Abstract

Bound state solutions of the Schrödinger equation have been investigated for n-dimensional ($n \geqslant 2$) harmonic oscillator potential decorated with any finite number (P) of Dirac delta functions. The potential is radially symmetric and given as $V(r) = \frac{1}{2}m\omega^2 r^2 - \frac{\hbar^2}{2m}\sum_{i=1}^{P}\sigma_i\delta(r - r_i)$, where σ_is are arbitrary real numbers, $r_1 < r_2 < \cdots < r_P$ and $r_i \in (0, +\infty)$. We have demonstrated that addition of Dirac delta functions lifts the accidental degeneracies of n-dimensional harmonic oscillator energy levels and leaves only the degeneracy due to the radial symmetry. Explicit forms of bound state eigenfunctions and the eigenvalue equation are given for n, l values, where n is the space dimension and l is the degree of n-dimensional spherical harmonics. We have shown that, for given n and l, there are a countably infinite number of bound state energy levels which are continuous functions of ω, σ_is and at most P of them can be negative.

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1. Introduction

A quantum harmonic oscillator in any dimension is exactly solvable, and its solutions are used to describe many important physical processes such as modes of radiation, phonons, molecular vibrations, etc [1, 2]. The Schrödinger equation for a particle of mass m in a potential with Dirac delta functions is also frequently studied in quantum mechanics [2–5]. Solutions of the Schrödinger equation with Dirac delta functions can be valuable for the description of extremely short-range or contact (point) interactions [4].

Spectral properties of the Schrödinger equation for harmonic oscillator potential together with a point interaction have been investigated by several authors [6–14]. The spectrum of the one-dimensional harmonic oscillator with one Dirac delta function has been studied in [6–9]. Two-dimensional systems in a uniform external magnetic field in the presence of a

δ-impurity and a cylindrical δ-potential have been investigated in [10] and [11] respectively. Spectral properties of three-dimensional harmonic oscillator with a point interaction have been considered in [12–14]. Spectral analysis of interactions supported by manifolds with codimension one has also been done in [5]. Non-relativistic quantum mechanical sphere interactions were investigated in [15–22]. In this paper, we study the properties of bound state solutions of the Schrödinger equation for n-dimensional ($n \geqslant 2$) harmonic oscillator potential together with radially symmetric Dirac delta interactions and present some explicit results for these bound state eigenvalues and eigenfunctions.

Magnetostatic traps used in experiments of boson gases and two-dimensional electron gas in a uniform magnetic field are two important examples which can be described by a harmonic potential. Ultra-thin quantum wells or impurities in these systems can be modelled by using Dirac delta functions. Two-dimensional systems which contain Dirac delta interaction on a closed loop together with a magnetic field for a charge particle have been studied in [23, 24]. A model with harmonic oscillator potential decorated with any finite number of radially symmetric Dirac delta functions at arbitrary non-zero radii can be utilized to describe contact interactions of a particle with some materials on concentric spherical shells or circular structures in a confining harmonic potential. Thus, our calculations can be useful in finding the changes in the harmonic oscillator spectrum stemming from these very short-range interactions. For an interaction $V_{\text{Sep.}} = \frac{m}{2}\sum_{i=1}^{n}\omega_i^2 x_i^2 - \frac{\hbar^2}{2m}\sum_{i=1}^{n}\sigma_i\delta(x_i - a_i)$, with $\vec{r} = (x_1, x_2, \ldots, x_n)$, the problem can be reduced to n independent one-dimensional systems. The methodology of this paper can be used to obtain similar results (and related bound state eigenfunctions) for a one-dimensional harmonic oscillator with a finite number of Dirac delta functions [25]. The statements (a), (b) and (c) of theorem 1 of this paper are also true for the one-dimensional case with analogous conditions.

2. Results and discussion

Our aim is to obtain and analyse bound state solutions of the Schrödinger equation for n-dimensional ($n \geqslant 2$) oscillator potential together with P Dirac delta functions. The potential is radially symmetric and given as

$$V(r) = \frac{1}{2}m\omega^2 r^2 - \frac{\hbar^2}{2m}\sum_{i=1}^{P}\sigma_i\delta(r - r_i), \tag{1}$$

where $\omega > 0, \sigma_i$ are arbitrary real numbers and $r_1 < r_2 < \cdots < r_P$ with $r_i \in (0, +\infty)$. The factor $-\left(\frac{\hbar^2}{2m}\right)$ is for calculational convenience. Negative σ_i value represents repulsive interaction while positive σ_i value represents attractive interaction.

The time-independent Schrödinger equation for a particle with mass m in the potential $V(x)$ is given as

$$\mathcal{H}_{\omega\sigma}\Psi(x_1, \ldots, x_n) = \left(-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right)\Psi(x_1, \ldots, x_n) = E\Psi(x_1, \ldots, x_n), \tag{2}$$

where $\nabla^2 = \sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}$.

Since the potential depends only on r, we write the wavefunction in terms of spherical coordinates as $\Psi = R_{n,l}(r)Y_{l,n}(\Theta)$, where $Y_{l,n}(\Theta)$ is an n-dimensional spherical harmonic of degree l and $\Theta = (\theta_1, \ldots, \theta_{n-1})$ represents $n - 1$ angular coordinates [26]. Then, the Laplacian in spherical coordinates becomes

$$\nabla^2 = \frac{1}{r^{n-1}}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^{n-1}\frac{\mathrm{d}(.)}{\mathrm{d}r}\right) + \frac{\Omega_{\text{LB}}}{r^2}, \tag{3}$$

where the Laplace–Beltrami operator, Ω_{LB}, on the sphere, $\mathbf{S}^{n-1}$, satisfies

$$\Omega_{\mathrm{LB}} Y_{l,n}(\Theta) = -l(l+n-2) Y_{l,n}(\Theta), \tag{4}$$

for $l = 0, 1, 2, \ldots$. The degeneracy of $Y_{l,n}$ is $m_{l,n} = \frac{(2l+n-2)(l+n-3)!}{l!(n-2)!}$.[1] By using μ index for these degenerate states, we take orthonormal set $\{Y_{l,n;\mu}\}$, for $\mu = 1, 2, \ldots, m_{l,n}$, that is

$$\int_{S^{n-1}} Y^*_{l,n;\mu} Y_{l,n;\nu} \, \mathrm{d}\Theta = \delta_{\mu,\nu} \tag{5}$$

[26].

For all σ_is are zero, we have n-dimensional harmonic oscillator with energies $E[q, l] = E[K] = (2q + l + \frac{n}{2})\hbar\omega = (K + \frac{n}{2})\hbar\omega$, where $K = 0, 1, 2, \ldots$ and q is the number nodes of the eigenfunction with energy $E[K]$. By using the same form of the harmonic oscillator spectrum, we define $E = (\Lambda + \frac{n}{2})\hbar\omega$, where $\Lambda = \Lambda(\omega, \sigma_1, \sigma_2, \ldots, \sigma_p, r_1, r_2, \ldots, r_p)$ is a real function of $\omega, \sigma_1, \sigma_2, \ldots, r_1, r_2, \ldots, r_p$. In general, Λ may not be non-negative integer for non-zero σ_i values. In theorem 1, we will show that they can be even negative real numbers for some lowest eigenvalues. Variations of $r_1, r_2, \ldots, r_p$ will change the locations of delta functions. We fix $r_1, r_2, \ldots, r_p$ values and analyse the energy eigenvalues as a function of $\omega, \sigma_1, \sigma_2, \ldots, \sigma_p$.

By inserting $E = (\Lambda + \frac{n}{2})\hbar\omega$, we have the radial equation

$$-\frac{\hbar^2}{2m}\left\{\frac{1}{r^{n-1}}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^{n-1}\frac{\mathrm{d}(R_{n,l}(r))}{\mathrm{d}r}\right) - \frac{l(l+n-2)}{r^2}\right\} R_{n,l}(r) + \frac{1}{2}m\omega^2 r^2 R_{n,l}(r)$$
$$-\frac{\hbar^2}{2m}\left\{\sum_{i=1}^{P}\sigma_i\delta(r-r_i)\right\} R_{n,l}(r) = \left(\Lambda + \frac{n}{2}\right)\hbar\omega R_{n,l}(r), \tag{6}$$

for $l = 0, 1, 2, \ldots$ and $n \geqslant 2$.

By using dimensionless parameter $v = \sqrt{\frac{2m\omega}{\hbar}} r$, we obtain

$$\frac{1}{v^{n-1}}\frac{\mathrm{d}}{\mathrm{d}v}\left(v^{n-1}\frac{\mathrm{d}(R_{n,l}(v))}{\mathrm{d}v}\right) - \frac{l(l+n-2)}{v^2} R_{n,l}(v) + \left\{\left(\Lambda + \frac{n}{2}\right) - \frac{v^2}{4}\right\} R_{n,l}(v)$$
$$+\left\{\sum_{i=1}^{P}\zeta_i\delta(v-v_i)\right\} R_{n,l}(v) = 0, \tag{7}$$

where $v_i = \sqrt{\frac{2m\omega}{\hbar}} r_i$ and $\zeta_i = \frac{\sigma_i}{\sqrt{\frac{2m\omega}{\hbar}}}$ for $i = 1, 2, \ldots, P$.

When $v \neq v_i$, equation (7) reduces to

$$\frac{1}{v^{n-1}}\frac{\mathrm{d}}{\mathrm{d}v}\left(v^{n-1}\frac{\mathrm{d}(R_{n,l}(v))}{\mathrm{d}v}\right) - \frac{l(l+n-2)}{v^2} R_{n,l}(v) + \left\{\left(\Lambda + \frac{n}{2}\right) - \frac{v^2}{4}\right\} R_{n,l}(v) = 0. \tag{8}$$

Equation (8) has two linearly independent solutions. By trying solutions of the form $R_{n,l}(v) = v^{\rho} \, \mathrm{e}^{-\frac{v^2}{4}} u(v)$, we get solutions

$$\varphi_A = v^l \, \mathrm{e}^{-\frac{v^2}{4}} \psi\left(\frac{l-\Lambda}{2}, l + \frac{n}{2}; \frac{v^2}{2}\right) \tag{9a}$$

and

$$\varphi_B = v^l \, \mathrm{e}^{-\frac{v^2}{4}} \phi\left(\frac{l-\Lambda}{2}, l + \frac{n}{2}; \frac{v^2}{2}\right), \tag{9b}$$

[1] $m_{0,2} = 1$. This can also be obtained from the general formula by first inserting $l = 0$, doing cancellations and then inserting n.

where $\phi(\alpha, \gamma; z)$ is the confluent hypergeometric function of the first kind and $\psi(\alpha, \gamma; z) = \frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)}\phi(\alpha, \gamma; z) + \frac{\Gamma(\gamma-1)}{\Gamma(\alpha)} z^{1-\gamma}\phi(1+\alpha-\gamma, 2-\gamma; z)$ is the confluent hypergeometric function of the second kind. Throughout this work,$\alpha = \frac{l-\Lambda}{2}$, $\gamma = l + \frac{n}{2}$ and $z = \frac{v^2}{2}$. Since $\phi(\alpha, \gamma; z)$ and $\psi(\alpha, \gamma; z)$ are entire functions of α, $\gamma > 0$ and $z > 0$, the solutions φ_A and φ_B are also entire functions of v, l, n. (For $z = 0$, $\phi(\alpha, \gamma; 0) = 1$. By using analytic continuation, ψ is defined for integer γ values [27].) When α is a non-positive integer, $\Gamma(\alpha)$ becomes infinite and φ_A, φ_B are now linearly dependent. Then, linearly independent solutions of equation (8) will be constructed in terms of Laguerre polynomials. We will later show explicit form of two independent solutions of equation (8) when $\alpha = 0, -1, -2, \ldots$.

We first consider the case for $\alpha \neq 0, -1, -2, \ldots$. By taking $v_0 = 0$ and $v_{P+1} = +\infty$, we define ith interval as $[v_{i-1}, v_i]$, for $i = 1, 2, \ldots, P+1$. Then, for given n and l, the general solution of equation (7) is

$$R_{n,l}(v) = a_i\varphi_A(v) + b_i\varphi_B(v) \quad \text{when} \quad v \in [v_{i-1}, v_i] \quad \text{and} \quad i = 1, 2, \ldots, P+1. \tag{10}$$

For large z values, we have $\phi(\alpha, \gamma; z) \approx \frac{\Gamma(\gamma)}{\Gamma(\alpha)} \mathrm{e}^z z^{-(\gamma-\alpha)}$ and $\psi(\alpha, \gamma; z) \approx z^{-\alpha}$. For $z \to 0$ and positive γ, $\phi(\alpha, \gamma; z) \to 1$ and $\psi(\alpha, \gamma; z) \to \frac{\Gamma(\gamma-1)}{\Gamma(\alpha)} z^{1-\gamma}$ when $\gamma \neq 1$ or $\psi(\alpha, \gamma; z) \to -\frac{\ln(z)}{\Gamma(\alpha)}$ when $\gamma = 1$ [27]. Thus, for $z = \frac{v^2}{2}$, we find $\varphi_A \to +\infty$ as $v \to 0$, and $\varphi_B \to +\infty$ as $v \to +\infty$. Hence, we have to take $a_1 = 0$ and $b_{P+1} = 0$ which lead to $b_1\varphi_B(v)$ for the first interval and $a_{P+1}\varphi_A(v)$ for the $(P+1)th$ interval as the regular solutions of equation (7). Since $\psi\big(\alpha, \gamma; \frac{v^2}{2}\big) \approx \big(\frac{v^2}{2}\big)^{-\alpha}$ for large v values, $R_{n,l}$ decays exponentially as $v \to +\infty$. Hence, the integral $I = \int_0^\infty (R_{n,l}(r))^2 r^{n-1}\,\mathrm{d}r = \int_0^\infty (R_{n,l}(v))^2 \big(\frac{\hbar}{2m\omega}\big)^{\frac{n}{2}} v^{n-1}\,\mathrm{d}v$ converges. Thus, bound states Ψ with radial part of which is given in equation (10) are normalizable.

The continuity of the wavefunction at the boundary of ith and $(i+1)th$ intervals leads to

$$a_i\varphi_A(v_i) + b_i\varphi_B(v_i) = a_{i+1}\varphi_A(v_i) + b_{i+1}\varphi_B(v_i). \tag{11}$$

By multiplying equation (7) with $v^{n-1}\mathrm{d}v$, we integrate these equations between $v_i - \epsilon$ and $v_i + \epsilon$. Letting $\epsilon \to 0^+$, using the continuity of the wavefunctions and cancelling v_i^{n-1} terms, we get

$$(a_{i+1}\varphi_A'(v_i) + b_{i+1}\varphi_B'(v_i)) - (a_i\varphi_A'(v_i) + b_i\varphi_B'(v_i)) + \zeta_i(a_i\phi_A(v_i) + b_i\phi_B(v_i)) = 0, \tag{12}$$

where $'$ denotes differentiation with respect to v. By solving linear equations (11) and (12) for a_{i+1} and b_{i+1} in terms of a_i and b_i, we obtain the recursion relations

$$\begin{aligned} a_{i+1} &= \left(1 + \frac{\zeta_i\varphi_A(v_i)\varphi_B(v_i)}{W_i}\right)a_i + \left(\frac{\zeta_i(\varphi_B(v_i))^2}{W_i}\right)b_i, \\ b_{i+1} &= \left(-\frac{\zeta_i(\varphi_A(v_i))^2}{W_i}\right)a_i + \left(1 - \frac{\zeta_i\varphi_A(v_i)\varphi_B(v_i)}{W_i}\right)b_i, \end{aligned} \tag{13}$$

where $W_i = W_i[\varphi_A, \varphi_B] = \varphi_A(v_i)\varphi_B'(v_i) - \varphi_B(v_i)\varphi_A'(v_i)$ is the Wronskian.

We define the transfer matrix M_i, and write equation (13) in terms of M_i:

$$\begin{pmatrix} a_{i+1} \\ b_{i+1} \end{pmatrix} = M_i \begin{pmatrix} a_i \\ b_i \end{pmatrix} = \begin{pmatrix} 1 + \frac{\zeta_i\varphi_A(v_i)\varphi_B(v_i)}{W_i} & \frac{\zeta_i(\varphi_B(v_i))^2}{W_i} \\ -\frac{\zeta_i(\varphi_A(v_i))^2}{W_i} & 1 - \frac{\zeta_i\varphi_A(v_i)\varphi_B(v_i)}{W_i} \end{pmatrix} \begin{pmatrix} a_i \\ b_i \end{pmatrix}. \tag{14}$$

Thus,

$$\begin{pmatrix} a_{P+1} \\ b_{P+1} \end{pmatrix} = M_P M_{P-1} \cdots M_1 \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \mathbf{X}\begin{pmatrix} a_1 \\ b_1 \end{pmatrix}, \tag{15}$$

where the matrix $\mathbf{X} = \left(\begin{smallmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{smallmatrix}\right) = M_P M_{P-1} \cdots M_1$ is a function of Λ. Since we demand $a_1 = 0$ and $b_{P+1} = 0$ for regular solutions, then, for given n and l, we obtain $\mathbf{X}_{22}(\Lambda) = 0$

which is a transcendental equation in general. The roots of the equation $\mathbf{X}_{22}(\Lambda) = 0$ will be used to find the energy levels, $E = \left(\Lambda + \frac{n}{2}\right)\hbar\omega$.

For φ_A and φ_B, $W[\varphi_A, \varphi_B] = \frac{2^{\gamma}\Gamma(\gamma)}{\Gamma(\alpha)v^{n-1}}$, we have

$$M_i = \begin{pmatrix} 1 + \frac{\zeta_i \phi(\alpha,\gamma;z_i)\psi(\alpha,\gamma;z_i)}{\frac{\Gamma(\gamma)}{\Gamma(\alpha)}\sqrt{2}z_i{}^{-(\gamma-\frac{1}{2})}\,\mathrm{e}^{z_i}} & \frac{\zeta_i[\phi(\alpha,\gamma;z_i)]^2}{\frac{\Gamma(\gamma)}{\Gamma(\alpha)}\sqrt{2}z_i{}^{-(\gamma-\frac{1}{2})}\,\mathrm{e}^{z_i}} \\ -\frac{\zeta_i[\psi(\alpha,\gamma;z_i)]^2}{\frac{\Gamma(\gamma)}{\Gamma(\alpha)}\sqrt{2}z_i{}^{-(\gamma-\frac{1}{2})}\,\mathrm{e}^{z_i}} & 1 - \frac{\zeta_i \phi(\alpha,\gamma;z_i)\psi(\alpha,\gamma;z_i)}{\frac{\Gamma(\gamma)}{\Gamma(\alpha)}\sqrt{2}z_i{}^{-(\gamma-\frac{1}{2})}\,\mathrm{e}^{z_i}} \end{pmatrix}, \tag{16}$$

where $z_i = \frac{v_i^2}{2}$. Thus, by solving $\mathbf{X}_{22}(\Lambda) = 0$, we obtain Λ and hence M_is which in turn determine $R_{n,l}$ exactly. For a special case $P = 1$, we have

$$\mathbf{X}_{22}(\Lambda) = 1 - \frac{\zeta_1 \phi(\alpha, \gamma; z_1)\psi(\alpha, \gamma; z_1)}{\frac{\Gamma(\gamma)}{\Gamma(\alpha)}\sqrt{2}z_1{}^{-(\gamma-\frac{1}{2})}\,\mathrm{e}^{z_1}} = 0. \tag{17}$$

Then, we get

$$\phi\left(\frac{l-\Lambda}{2}, \gamma; z_1\right)\psi\left(\frac{l-\Lambda}{2}, \gamma; z_1\right)\Gamma\left(\frac{l-\Lambda}{2}\right) = \frac{\Gamma(\gamma)\sqrt{2}z_1{}^{-(\gamma-\frac{1}{2})}\,\mathrm{e}^{z_1}}{\zeta_1}, \tag{18}$$

which will be solved for Λ for given n, l, ω, σ_1 and r_1. For $n = 3, l = 0, \sigma_1 = \frac{\Gamma\left(\frac{3}{2}\right)\sqrt{\frac{m\omega}{\hbar}}\,2\mathrm{e}}{\phi\left(\frac{3}{4},\frac{3}{2};1\right)\psi\left(\frac{3}{4},\frac{3}{2};1\right)\Gamma\left(\frac{3}{4}\right)}$ and $r_1 = \sqrt{\frac{\hbar}{m\omega}}$, one of the solutions of equation (18) is $\Lambda = -\frac{3}{2}$ and hence $E = 0$. For $\sigma_1 = \frac{\Gamma\left(\frac{3}{2}\right)\sqrt{\frac{m\omega}{\hbar}}\,2\mathrm{e}}{\phi\left(\frac{5}{4},\frac{3}{2};1\right)\psi\left(\frac{5}{4},\frac{3}{2};1\right)\Gamma\left(\frac{5}{4}\right)}$, we get $\Lambda = -\frac{5}{2}$ and hence $E = -\hbar\omega$. In theorem 1, we will demonstrate that there are infinitively many solutions of $X_{22}(\Lambda) = 0$, and there can be at most P negative eigenvalues for P Dirac delta functions.

For $z_i \gg 1$ and $z_i \gg |\zeta_i|$, we can obtain the approximate change in the energies by using asymptotic behaviours of confluent hypergeometric functions and the equation $X_{22}(\Lambda) = 0$ or performing the first-order perturbation for the harmonic oscillator potential with the perturbing Dirac delta interactions. Both methods give the same result. As an example, for $z_i \gg 1$ and $z_i \gg |\zeta_i|, n = 3, l = 0$, we have

$$\Delta E[q, l=0] = E[q, l=0] - \left(2q + \frac{3}{2}\right)\hbar\omega \approx \sum_{i=1}^{P} -2\hbar\omega\zeta_i \frac{z_i^{2q+1}}{\sqrt{2}q!\Gamma\left(q+\frac{3}{2}\right)}\,\mathrm{e}^{-z_i}$$

$$\approx -\left(\frac{m\omega}{\hbar}\right)^{2q+\frac{1}{2}} \frac{\hbar\omega}{q!\Gamma\left(q+\frac{3}{2}\right)} \sum_{i=1}^{P} \sigma_i r_i^{4q+2}\,\mathrm{e}^{-\frac{m\omega}{\hbar}r_i^2}. \tag{19}$$

For intermediate values of $|\zeta_i|$ and z_i or $|\zeta_i| \gg z_i$, asymptotic of confluent hypergeometric functions or the first-order perturbation does not supply the approximate solutions of $X_{22}(\Lambda) = 0$, and numerical solutions of $X_{22}(\Lambda) = 0$ should be obtained.

In tables 1 and 2, we present numerical results of some quantities that are related to the ground state and the first excited state energies for large attractive and repulsive ζ_i values, i.e. $\zeta_i = \pm 10.0$. For one delta function case with $|\zeta_1| = +10.0$, as seen in the fifth row of table 1, the absolute change in the ground state energy is larger for attractive(A) Dirac interactions than for repulsive(R) delta interactions, i.e. $\frac{|\Delta E_{g,A}|}{|\Delta E_{g,R}|} > 1$ where $|\Delta E_x| = |E_x(\zeta_1 \neq 0.0) - E_x(\zeta_1 = 0.0)|$. However, for the first excited state, $\frac{|\Delta E_{2,A}|}{|\Delta E_{2,R}|}$ is larger than 1 (for $z_1 = 2.0, 10.0$) or smaller than 1 (for $z_1 = 0.5, 1.0$), depending on the location (or z_1) of the delta function. By using tables 1 and 2, one gets the following orderings for attractive and repulsive interactions with $|\zeta_1| = +10.0$:

Table 1. Some dimensionless quantities which are related to energies for one Dirac delta function with strength $\zeta_1 = +10.0$ (attractive(A) case) or $\zeta_1 = -10.0$ (repulsive(R) case) at $r = r_1$ ($z_1 = \frac{m\omega}{\hbar} r_1^2$). Here $n = 3, l = 0$ and E_g and E_2 denote the ground state and the first excited state energies respectively. $|\Delta E_x| = |E_x(\zeta_1 \neq 0.0) - E_x(\zeta_1 = 0.0)|$, where $E_g(\zeta_1 = 0.0) = \frac{3}{2}$ and $E_2(\zeta_1 = 0.0) = 2 + \frac{3}{2}$.

	$z_1 = 0.5$	$z_1 = 1.0$	$z_1 = 2.0$	$z_1 = 10.0$
$\left(\frac{E_{g,A}}{\hbar\omega} - \frac{3}{2}\right)$	−26.242	−25.996	−25.496	−21.497
$\left(\frac{E_{g,R}}{\hbar\omega} - \frac{3}{2}\right)$	0.884	1.380	1.038	0.002
$\left(\frac{E_{2,A}}{\hbar\omega} - \frac{3}{2}\right)$	1.122	1.654	1.460	0.004
$\left(\frac{E_{2,R}}{\hbar\omega} - \frac{3}{2}\right)$	3.220	2.956	2.296	2.074
$\frac{\lvert\Delta E_{g,A}\rvert}{\lvert\Delta E_{g,R}\rvert}$	29.69	18.84	25.56	10750
$\frac{\lvert\Delta E_{2,A}\rvert}{\lvert\Delta E_{2,R}\rvert}$	0.7197	0.3619	1.824	26.97

Table 2. Some dimensionless quantities which are related to energies for two Dirac delta functions with strengths $\zeta_i = +10.0$ (attractive(A) case) or $\zeta_i = -10.0$ (repulsive(R) case) at $r = r_i$ ($z_i = \frac{m\omega}{\hbar} r_i^2$). Here $n = 3$, $l = 0$ and E_g and E_2 denote the ground and the first excited state energies respectively. $|\Delta E_x| = |E_x(\zeta_1 \neq 0.0, \zeta_2 \neq 0.0) - E_x(\zeta_1 = 0.0, \zeta_2 = 0.0)|$, where $E_g(\zeta_1 = 0.0, \zeta_2 = 0.0) = \frac{3}{2}$ and $E_2(\zeta_1 = 0.0, \zeta_2 = 0.0) = 2 + \frac{3}{2}$.

	$(z_1 = 1.0, z_2 = 2.0)$	$(z_1 = 1.0, z_2 = 4.0)$	$(z_1 = 1.0, z_2 = 10.0)$
$\left(\frac{E_{g,A}}{\hbar\omega} - \frac{3}{2}\right)$	−28.145	−25.996	−25.995
$\left(\frac{E_{g,R}}{\hbar\omega} - \frac{3}{2}\right)$	2.272	2.692	1.418
$\left(\frac{E_{2,A}}{\hbar\omega} - \frac{3}{2}\right)$	−22.604	−24.494	−21.490
$\left(\frac{E_{2,R}}{\hbar\omega} - \frac{3}{2}\right)$	3.000	3.566	2.984
$\frac{\lvert\Delta E_{g,A}\rvert}{\lvert\Delta E_{g,R}\rvert}$	12.39	9.657	18.33
$\frac{\lvert\Delta E_{2,A}\rvert}{\lvert\Delta E_{2,R}\rvert}$	24.60	16.92	23.87

(a) $\left|\Delta E_{g,A,z_1=0.5}\right| > \left|\Delta E_{g,A,z_1=1.0}\right| > \left|\Delta E_{g,A,z_1=2.0}\right| > \left|\Delta E_{g,A,z_1=10.0}\right|$
(b) $\left|\Delta E_{g,R,z_1=1.0}\right| > \left|\Delta E_{g,R,z_1=2.0}\right| > \left|\Delta E_{g,R,z_1=0.5}\right| > \left|\Delta E_{g,R,z_1=10.0}\right|$
(c) $\left|\Delta E_{2,A,z_1=10.0}\right| > \left|\Delta E_{2,A,z_1=0.5}\right| > \left|\Delta E_{2,A,z_1=2.0}\right| > \left|\Delta E_{2,A,z_1=1.0}\right|$
(d) $\left|\Delta E_{2,R,z_1=0.5}\right| > \left|\Delta E_{2,R,z_1=1.0}\right| > \left|\Delta E_{2,R,z_1=2.0}\right| > \left|\Delta E_{2,R,z_1=10.0}\right|$
for one delta function, and
(e) $\left|\Delta E_{g,A,(z_1=1.0,z=2.0)}\right| > \left|\Delta E_{g,A,(z_1=1.0,z=4.0)}\right| > \left|\Delta E_{g,A,(z_1=1.0,z=10.0)}\right|$
(f) $\left|\Delta E_{g,R,(z_1=1.0,z=4.0)}\right| > \left|\Delta E_{g,R,(z_1=1.0,z=2.0)}\right| > \left|\Delta E_{g,R,(z_1=1.0,z=10.0)}\right|$
(g) $\left|\Delta E_{2,A,(z_1=1.0,z=4.0)}\right| > \left|\Delta E_{2,A,(z_1=1.0,z=2.0)}\right| > \left|\Delta E_{2,A,z_1=1.0,z=10.0)}\right|$
(h) $\left|\Delta E_{2,R,(z_1=1.0,z=4.0)}\right| > \left|\Delta E_{g,R,(z_1=1.0,z=2.0)}\right| > \left|\Delta E_{g,R,(z_1=1.0,z=10.0)}\right|$

for two delta functions. Thus, these numerical results demonstrate that in general the change in the energy levels has a complicated dependence on the positions of Dirac delta functions.

As we vary σ_i values, α can take non-positive integer values for some eigenfunctions. Then, φ_A and φ_B are linearly dependent. For $\alpha = \frac{l-\Lambda}{2} = -q = 0, -1, -2, \ldots,$ we will construct the solution $R_{n,l}$ in terms of Laguerre polynomials, $L_q^{\gamma-1}$ by using the relations $\phi(-q,\gamma;z)=\frac{q!}{(\gamma)(\gamma+1)\cdots(\gamma+q-1)}L_q^{\gamma-1}(z)$ if $q\neq 0$ and $\phi(0,\gamma;z)=L_0^{\gamma-1}(z)=1$ if $q=0$. For this case, one of the solutions of equation (8) is $\bar{\varphi}_A=v^l\,\mathrm{e}^{-\frac{v^2}{4}}L_q^{\gamma-1}\big(\frac{v^2}{2}\big)$. For n odd integer, the second solution is $\bar{\varphi}_B=v^l\,\mathrm{e}^{-\frac{v^2}{4}}\big(\frac{v^2}{2}\big)^{1-\gamma}\phi(1+\alpha-\gamma,2-\gamma;\frac{v^2}{2})$. Then, for n odd integer,

$$\begin{aligned}R_{nl}&=\bar{a}_i\bar{\varphi}_A+\bar{b}_i\bar{\varphi}_B\\&=\bar{a}_i\left\{v^l\,\mathrm{e}^{-\frac{v^2}{4}}L_q^{\gamma-1}\left(\frac{v^2}{2}\right)\right\}+\bar{b}_i\left\{v^l\,\mathrm{e}^{-\frac{v^2}{4}}\phi\left(1+\alpha-\gamma,2-\gamma;\frac{v^2}{2}\right)\right\}.\end{aligned}\tag{20}$$

For n even integer, the indicial equation for the differential equation (8) has integer roots, and the second solution has the form $\bar{\varphi}_B=v^l\,\mathrm{e}^{-\frac{v^2}{4}}\big\{g_sL_q^{\gamma-1}\big(\frac{v^2}{2}\big)\ln\big(\frac{v^2}{2}\big)+\big(\frac{v^2}{2}\big)^{1-\gamma}\sum_{i=0}^{\infty}b_n\big(\frac{v^2}{2}\big)^n\big\}$, where the coefficients g_s and b_n are found by inserting the expression in Laguerre differential equation [28].

By following the similar procedure which leads to equations (11) and (12), we obtain the recursion relations

$$\begin{pmatrix}\bar{a}_{i+1}\\\bar{b}_{i+1}\end{pmatrix}=\bar{M}_i\begin{pmatrix}\bar{a}_i\\\bar{b}_i\end{pmatrix}=\begin{pmatrix}1+\frac{\zeta_i\bar{\varphi}_A(v_i)\bar{\varphi}_B(v_i)}{\bar{W}_i}&\frac{\zeta_i(\bar{\varphi}_B(v_i))^2}{\bar{W}_i}\\-\frac{\zeta_i(\bar{\varphi}_A(v_i))^2}{\bar{W}_i}&1-\frac{\zeta_i\bar{\varphi}_A(v_i)\bar{\varphi}_B(v_i)}{\bar{W}_i}\end{pmatrix}\begin{pmatrix}\bar{a}_i\\\bar{b}_i\end{pmatrix},$$

where $\bar{W}_i[\bar{\varphi}_A,\bar{\varphi}_B]=\frac{\Gamma(\gamma-\alpha)(1-\gamma)}{\Gamma(\gamma)\Gamma(-\alpha+1)}\frac{1}{v^{n-1}}$, for $\gamma\neq 1,2,\ldots$, $\bar{W}_i=\frac{1}{v^{n-1}}$, for $\gamma=1$, and $\bar{W}_i=\frac{\Gamma(\gamma-\alpha)}{\Gamma(\gamma)\Gamma(-\alpha+1)(1-\gamma)}\frac{1}{v^{n-1}}$, for $\gamma=2,3,\ldots$.

We note that, given n and l, the solutions $R_{n,l}(v)$ of equation (7) are non-degenerate. This result can be obtained by using transfer matrices (theorem 1 in [29] demonstrates such a derivation). An alternative demonstration of this result and ordering of the corresponding eigenvalues can be deduced by using the results of Hilbert–Courant vol I [30]. Since $R_{n,l}(v)$s are continuous and have piecewise continuous first derivatives (with finitely many finite amount of jumps), the requirements for the applicability of the maximum–minimum property in [30] are satisfied. By applying the results of chapter VI of Hilbert–Courant, we get that, given n and l, the solutions $R_{n,l}(v)$ of equation (7) are non-degenerate and corresponding eigenvalues (energies) can be ordered such that the eigenvalue (energy) is higher for a bound state with larger number of nodes. Then, by taking q as node index, we order the energies, that is

$$E_0<E_1<\cdots<E_q<\cdots.\tag{21}$$

As an example, given $n, l=0$ (s state), by using the spectroscopic notation $E_{N\mathrm{s}}\equiv E[q=N-1,l=0]$, the order of the levels is found as $E_{1\mathrm{s}}<E_{2\mathrm{s}}<\cdots<E_{N\mathrm{s}}<\cdots$. By defining $f_q=E_q(\omega,\sigma_1,\sigma_2,\ldots,\sigma_p)$ and taking $(\omega,\sigma_1,\sigma_2,\ldots,\sigma_p,f_q)$ as coordinates, we obtain a surface in $\mathbf{R}^{P+2}$ for each eigenvalue with node index q.

For the harmonic oscillator, all bound state energies are positive. We have seen that some of the eigenvalues of $\mathcal{H}_{\omega\sigma}$ can be zero or negative. We will prove theorem 1 about the properties of bound state energies of the Hamiltonian $\mathcal{H}_{\omega\sigma}=-\frac{\hbar^2}{2m}\nabla^2+\frac{1}{2}m\omega^2r^2-\frac{\hbar^2}{2m}\sum_{i=1}^{P}\sigma_i\delta(r-r_i)$ and compare with the bound state energies of $\mathcal{H}_0=-\frac{\hbar^2}{2m}\nabla^2-\frac{\hbar^2}{2m}\sum_{i=1}^{P}\sigma_i\delta(r-r_i)$.

Theorem 1. *For the Hamiltonian $\mathcal{H}_{\omega\sigma}$ with arbitrary real σ_i and arbitrary positive r_i values, for given n and l*

(a) bound state energies are continuous functions of $\omega,\sigma_1,\ldots,\sigma_P$,
(b) there exist countably infinite number of bound state eigenvalues,
(c) at most P eigenvalues for the bound states can be negative.

Proof. (a) For given n and l, we define $R_{n,l;q}$ as the radial part of the bound state eigenfunction of $\mathcal{H}_{\omega\sigma}$ with q nodes. By dropping n, l subscripts, we take $R_q \equiv R_{n,l;q}$. We define energy surface $f_q = E_q(\omega, \sigma_1, \ldots, \sigma_P)$ for the energy of bound state $\Psi = R_q(v)Y_{l,n;\mu}(\Theta)$. By Hellmann–Feynman theorem, for normalized wavefunction $\Psi\big(\sqrt{\frac{2m\omega}{\hbar}}r\big)$ of $\mathcal{H}_{\omega\sigma}$, we have

$$\frac{\partial E_q}{\partial\omega} = \langle\Psi|\frac{\partial\mathcal{H}_{\omega\sigma}}{\partial\omega}|\Psi\rangle = m\omega\int_\tau r^2\left|\Psi\left(\sqrt{\frac{2m\omega}{\hbar}}r\right)\right|^2 \mathrm{d}\tau$$

$$= m\omega\int_0^{+\infty} r^2\left|R_q\left(\sqrt{\frac{2m\omega}{\hbar}}r\right)\right|^2 r^{n-1}\,\mathrm{d}r, \tag{22}$$

where we have used orthonormality of $Y_{l,n;\mu}$ and $\mathrm{d}\tau$ represents n-dimensional 'volume element'. Since R_q of Ψ decays exponentially to zero as $r \to +\infty$, the integral in equation (22) converges. Thus, the derivative $\frac{\partial E_q}{\partial\omega}$ exists, and hence $f_q = E_q(\omega, \sigma_1, \ldots, \sigma_P)$ is a continuous function of ω. Similarly,

$$\frac{\partial E_q}{\partial\sigma_i} = \langle\Psi|\frac{\partial\mathcal{H}_{w\sigma}}{\partial\sigma_i}|\Psi\rangle = -\frac{\hbar^2}{2m}\left|R_q\left(\sqrt{\frac{2m\omega}{\hbar}}r_i\right)\right|^2 r_i^{n-1}. \tag{23}$$

Since $\frac{\partial E_q}{\partial\sigma_i}$ exists, $f_q = E_q(\omega, \sigma_1, \ldots, \sigma_P)$ is also a continuous function of σ_i for $i = 1, \ldots, P$.
(b) For given n and l, when $\sigma_i = 0$, for $i = 1, \ldots, P$, and $\omega = \omega_0$, we have the harmonic oscillator energies, $f_q^o = E_q(\omega_0, 0, \ldots, 0) = \big(2q + l + \frac{n}{2}\big)\hbar\omega_0$ for the node index $q = 0, 1, 2, \ldots$. Thus, we have countably infinite number of bound state solutions with these energies and $\big\{v^l\,\mathrm{e}^{-\frac{v^2}{4}}L_q^{\gamma-1}\big(\frac{v^2}{2}\big)Y_{l,n;\mu}\big\}$ as the eigenfunctions. In part (a), we have shown that E_q is a continuous function of $\omega, \sigma_1, \ldots, \sigma_P$. Then, starting from the point $\big(\omega_0, 0, \ldots, 0, f_q^o\big)$ and varying $\omega, \sigma_1, \ldots, \sigma_P$, we obtain a surface $f_q = E_q(\omega, \sigma_1, \ldots, \sigma_P)$ for each q where f_q is a continuous function of $(\omega, \sigma_1, \ldots, \sigma_P)$. Thus, for any given $\bar\omega, \bar\sigma_1, \ldots, \bar\sigma_P$, we have a point with coordinates $(\bar\omega, \bar\sigma_1, \ldots, \bar\sigma_P, \bar f_q)$ on each surface. For bound states, equation (21) leads to $E_\mathrm{s} > E_q$ if $\mathrm{s} > q$. Then, these surfaces do not intersect for given n and l. Hence, for any given $\omega, \sigma_1, \ldots, \sigma_P$, we have countably infinite number of energies E_q, where $q = 0, 1, 2, \ldots$.
(c) We take all σ_i positive, i.e. all the delta functions in the potential are attractive. With this choice, $\mathcal{H}_0$ will have at most P bound state solutions with the negative energies for given n and l [29]. Assume that we have N bound states of $\mathcal{H}_0$ with negative energies λ_j, where $0 < N \leqslant P$ and j represents the number of nodes of the corresponding radial part of the bound state eigenfunction.

For given n and l, we take R_j and T_j^o as the exact radial (bound state) eigenfunctions with j nodes of $\mathcal{H}_{w\sigma}$ and $\mathcal{H}_0$ respectively. Among the admissible functions F_j which satisfy boundary conditions and have j nodes, for a Hamiltonian $\mathcal{H}$, $\langle F_j|\mathcal{H}|F_j\rangle$ is the minimum for the exact eigenfunction [30]. Then, we have

$$\big\langle T_j^o\big|\mathcal{H}_0\big|T_j^o\big\rangle \leqslant \langle R_j|\mathcal{H}_0|R_j\rangle \leqslant \langle R_j|\mathcal{H}_{\omega\sigma} = \mathcal{H}_0 + \tfrac{1}{2}m\omega^2r^2|R_j\rangle \leqslant \big\langle T_j^o\big|\mathcal{H}_{\omega\sigma}\big|T_j^o\big\rangle, \tag{24}$$

which leads to

$$\lambda_j = E_j(0, \sigma_1, \ldots, \sigma_P) \leqslant E_j(\omega, \sigma_1, \ldots, \sigma_P) \leqslant E_j(0, \sigma_1, \ldots, \sigma_P) + \frac{1}{2}m\omega^2\int_0^{+\infty} r^2\big|T_j^o\big|^2 r^{n-1}\,\mathrm{d}r \tag{25}$$

Since $T_j^o \to c\,\mathrm{e}^{-\kappa r}$ as $r \to +\infty$, $\int_0^{+\infty}\big|T_j^o(r)\big|^2 r^{n+1}\,\mathrm{d}r$ converges [29]. Thus,

$$\lambda_j \leqslant E_j(\omega, \sigma_1, \ldots, \sigma_P) \leqslant \lambda_j + A\omega^2, \tag{26}$$

where $A = \frac{1}{2}m\int_0^{+\infty}\big|T_j^o\big|^2 r^{n+1}\,\mathrm{d}r$.

Thus, for sufficiently small ω or some particular values $\sigma_i > 0$ such that $\lambda_j < -A\omega^2$, we have

$$E_j(\omega; \sigma_1, \ldots, \sigma_P) < 0. \tag{27}$$

If j index of R_j with the corresponding negative energy is bigger than $P-1$, then equation (26) implies that there can be more than P bound states of $\mathcal{H}_0$ with negative energies. This is impossible since there exist at most P bound states of H_0 with negative energies [29]. Thus, there are at most P negative eigenvalues for the bound states of $\mathcal{H}_{\omega\sigma}$. The theorem is proven. □

For two-dimensional systems with Dirac delta interactions and a magnetic field (B), asymptotic behaviours of eigenvalues for large strengths (β) of delta functions have been investigated, and it was shown that the nth eigenvalue has an asymptotic form $\tilde{\lambda}_n = -\frac{\beta^2}{4} + \mu(B) + \mathcal{O}(\beta^{-1}\ln\beta)$ as $\beta \to \infty$ [23]. These results suggest that for sufficiently large strengths of delta interactions, there are exactly P negative eigenvalues of the Hamiltonian $\mathcal{H}_{\omega\sigma}$ of harmonic potential decorated with P attractive Dirac delta functions. This can be achieved by using the procedure of the min–max principle [32] and normalized wavefunctions $\Psi_q^{\text{h.o.}}(r)Y_{l,n}(\Theta)$ of harmonic oscillator and $R_q(r)Y_{l,n}(\Theta)$ of $\mathcal{H}_{\omega\sigma}$. For given n, l and ω, we first take the lowest eigenvalue harmonic oscillator wavefunction $\Psi_{q=0}^{\text{h.o.}}(r)Y_{l,n}(\Theta)$ as a trial function. Then, we have

$$E_0^{\text{tr}} = \langle \Psi_0^{\text{h.o.}} | \mathcal{H}_{\omega\sigma} | \Psi_0^{\text{h.o.}} \rangle = \left(l + \frac{n}{2}\right)\hbar\omega - \sum_{i=1}^{P} \sigma_i \left|\Psi_0^{\text{h.o.}}(x_i)\right|^2.$$

Hence, by increasing σ_is, we can make E_0^{tr} negative for the trial wavefunction $\Psi_0^{\text{h.o.}}$. Thus, for $\sigma_i > \sigma_{i,cr}$ (a critical value), we have at least one negative eigenvalue. We take the normalized radial wavefunction of the lowest eigenvalue as R_0. We continue the process of the min–max principle by using trial wavefunctions ψ_s^{tr} which are chosen to be square integrable functions and to have exactly s nodes and $\psi_s^{\text{tr}} \in [R_0, R_1, \ldots, R_{s-1}]^{\perp}$ for $s \geqslant 1$. (Here $[R_0, R_1, \ldots, R_{s-1}]^{\perp}$ is the notation for the set of wavefunctions which have the property $\langle \psi | R_j \rangle = 0$, for $j = 1, \ldots, s-1$, and R_j is the radial part of the eigenfunction of $\mathcal{H}_{\omega\sigma}$ with j nodes). Hence, we get

$$E_s^{\text{tr}} = \langle \psi_s^{\text{tr}} | H_{\text{har.osc.}} | \psi_s^{\text{tr}} \rangle - \sum_{i=1}^{P} \sigma_i \left|\psi_s^{\text{tr}}(x_i)\right|^2 = A_{\text{ho}} - \sum_{i=1}^{P} \sigma_i B_i,$$

where $A_{\text{ho}} = \langle \psi_s^{\text{tr}} | H_{\text{har.osc.}} | \psi_s^{\text{tr}} \rangle$ is a constant for given ψ_s^{tr} and $B_i = \left|\psi_s^{\text{tr}}(x_i)\right|^2$. ψ_s^{tr} cannot be zero for all x_i values ($i = 1, \ldots, P$) since ψ_s^{tr} has s nodes and $P > s$. Thus, at least some of B_is are non-zero and by choosing appropriate σ_is, we can make E_s^{tr} negative. Hence, by applying the min–max principle procedure, we have at least P negative eigenvalues for sufficiently large σ_is. Then, by combining this result with theorem 1(c), we obtain that there are exactly P negative eigenvalues of $\mathcal{H}_{\omega\sigma}$ with P attractive delta functions for sufficiently large strengths of Dirac delta functions.

Although energies of n-dimensional harmonic oscillator are not equal to each other for same l value, they are 'accidentally' equal to each other for some different l values. Energy level $E[K] = \left(K + \frac{n}{2}\right)\hbar\omega$ is $\binom{K+n-1}{n-1}$ times degenerate for $K = 0, 1, 2, \ldots$. By using angular degeneracy $m_{l,n}$ of Ψ, one can show that, for $n \geqslant 2$,

$$\binom{K+n-1}{n-1} = \begin{cases} \sum_{l=0,2,\ldots}^{K} m_{l,n} & \text{if} \quad K \text{ is an even integer} \\ \sum_{l=1,3,\ldots}^{K} m_{l,n} & \text{if} \quad K \text{ is an odd integer,} \end{cases} \tag{28}$$

where summations are taken over even or odd l values up to K. Thus, by using energy expressions for n-dimensional harmonic oscillator, the order of levels is

$$(1\text{s}), (1\text{p}), (1\text{d}, 2\text{s}), (1\text{f}, 2\text{p}), (1\text{g}, 2\text{d}, 3\text{s}), \ldots,$$

where the states in the same parentheses are 'accidentally' degenerate states of n-dimensional harmonic oscillator and the numbers in front of spectroscopic notation (s,l,d, . . .) are shell indices. The accidental degeneracy is due to an extra symmetry of the isotropic harmonic oscillator which is a quadratic function of p and r. $H_{\text{h.o.}} = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 r^2$ is invariant under any rotational transformations in n-dimensional space which are elements of n-dimensional orthogonal group $O(n)$(see $n = 3$ case in [31].) This Hamiltonian can be written as

$$H_{\text{h.o.}} = \sum_{i=1}^{n} \left(\hat{a}_i^\dagger \hat{a}_i + \frac{n}{2}\right)\hbar\omega,$$

where $\hat{a}_i^\dagger, \hat{a}_i$ are creation and annihilation operators respectively. This Hamiltonian will be invariant under any unitary transformations of $\hat{a}_i^\dagger, \hat{a}_i$ which are elements of n-dimensional unitary group $U(n)$. Since $U(n) \supset O(n)$, we have the extra symmetry of the harmonic oscillator Hamiltonian and accidental degeneracies occur. Only perturbations of the form $A_c p^2 + B_c r^2$ (A_c, B_c constants) can preserve this extra symmetry. Hence, by adding terms of the form $-\frac{\hbar^2}{2m}\sigma_i\delta(r - r_i)$, this extra symmetry is broken. Thus, the accidental degeneracy is lifted and only the rotational symmetry remains.

For $|\zeta_i| \ll 1$, the first-order perturbation contribution to energy is given as

$$\Delta E^{(1)} = -\left[\frac{q!}{\Gamma\left(q + l + \frac{n}{2}\right)}\right] \sqrt{2}\hbar\omega \sum_{i=1}^{P} \zeta_i z_i^{\gamma-\frac{1}{2}} \left[L_q^{\gamma-1}(z_i)\right]^2 \mathrm{e}^{-z_i}. \tag{29}$$

For one delta function and $|\zeta_1| \ll 1$, we have the perturbation expansion

$$\Delta E = \Delta E^{(1)} + \Delta E^{(2)} + \cdots + \Delta E^{(k)} + \cdots = a_1\zeta_1 + a_2\zeta_1^2 + \cdots + a_k\zeta_1^k + \cdots.$$

Then, for $n = 3$, $K = 2$, $z_1 = 1$ and $0 < \zeta_1 \ll 1$, by performing numerical calculations, we get $\Delta E_{1\text{d}}^{(1)} \neq 0$ and

$$\lim_{\zeta_1 \to 0} \frac{\Delta E_{2\text{s}}}{\Delta E_{1\text{d}}} = \frac{\Delta E_{2\text{s}}^{(1)}}{\Delta E_{1\text{d}}^{(1)}} = 0.6.$$

Thus, for this example, we have $\Delta E_{1\text{d}} < \Delta E_{2\text{s}}$ for sufficiently small positive ζ_1 (attractive delta function), and hence accidental degeneracy is lifted.

3. Conclusions

In this paper, we have analysed bound state solutions of the Schrödinger equation for n-dimensional ($n \geqslant 2$) harmonic oscillator potential decorated with any finite number (P) of Dirac delta functions. The potential is radially symmetric and given as $V(r) = \frac{1}{2}m\omega^2 r^2 - \frac{\hbar^2}{2m}\sum_{i=1}^{P}\sigma_i\delta(r - r_i)$, where σ_is are arbitrary real numbers, $r_1 < r_2 < \cdots < r_P$ and $r_i \in (0, +\infty)$ for $i = 1, 2, \ldots, P$. We have shown an explicit form of bound state eigenfunctions and obtained an equation for energies. We have demonstrated that addition of Dirac delta functions lifts the accidental degeneracies of n-dimensional harmonic oscillator energy levels and leaves only the degeneracy due to the radial symmetry. We have proved that for given n and l, there are countably infinite number of bound state energy levels which are continuous functions of ω, σ_is and at most P of them can be negative for the potential given above.

Contact (point) interactions of a particle in a harmonic confining potential with some impurities on concentric spherical shells or circular structures can be described by using the model that we have investigated. Our calculations can be used to find the changes in the harmonic oscillator spectrum due to these very short-range interactions.

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References

[1] Messiah A 1999 *Quantum Mechanics* (New York: Dover)
[2] Cohen-Tannoudji C, Diu B and Laloë F 1977 *Quantum Mechanics* vol I (Paris: Hermann)
[3] Gottfried K 1974 *Quantum Mechanics vol I: Fundamentals* (Reading, MA: WA Bejamin)
[4] Albeverio S, Gesztesy F, Høegh-Krohn R and Holden H 1988 *Solvable Models in Quantum Mechanics* (New York: Springer)
[5] Brasche J F, Exner P, Kuperin Yu A and Seba P 1994 *J. Math. Anal. Appl.* **184** 112–39
[6] Avakian M P, Pogosyan G S, Sissakian A N and Ter-Antonyan V M 1987 *Phys. Lett.* A **124** 233
[7] Fassari S and Inglese G 1994 *Helv. Phys. Acta* **67** 650–9
[8] Fassari S and Inglese G 1997 *Helv. Phys. Acta* **70** 858–65
[9] Olendski O 1995 *J. Phys.: Condens. Matter* **7** 5067–74
[10] Cavalcanti R M and de Carvalho C A A 1998 *J. Phys.: Math. Gen.* A **31** 2391–9
[11] Olendski O and Kim C S 1996 *J. Phys.: Condens. Matter* **8** 2197–208
[12] Janev R K and Maric Z 1974 *Phys. Lett.* **46** A 313–4
[13] Fassari S and Inglese G 1996 *Helv. Phys. Acta* **69** 130–40
[14] Brünig J, Geyler V and Lobanov I 2004 *J. Math. Phys.* **45** 1267–90
[15] Antoine J-P, Gesztesy F and Shabani J 1987 *J. Phys. A: Math. Gen.* **20** 3687–712
[16] Shabani J 1988 *J. Math. Phys.* **29** 660–4
[17] Shabani J 1988 *Nuovo Cimento* **101** B 429–37
[18] Dabrowski L and Shabani J 1988 *J. Math. Phys.* **29** 2241–4
[19] Antoine J-P, Exner P, Seba P and Shabani J 1994 *Ann. Phys.* **233** 1–16
[20] Hounkonnou M N, Hounkpe M and Shabani J 1997 *J. Math. Phys.* **38** 2832–50
[21] Hounkonnou M N, Hounkpe M and Shabani J 1999 *J. Math. Phys.* **40** 4255–73
[22] Shimada S 2003 *J. Math. Phys.* **44** 990–1005
[23] Exner P and Yoshitomi K 2002 *J. Phys. A: Math. Gen.* **35** 3479–87
[24] Exner P and Tater M 2004 *Waves Random Media* **14** S47–60
[25] Demiralp E Properties of One-Dimensional Harmonic Oscillator with Dirac Delta Functions unpublished notes
[26] Müller C 1966 *Spherical Harmonics Lecture Notes in Mathematics* (New York: Springer)
[27] Lebedev N N 1972 *Special Functions and Their Applications* (New York: Dover)
[28] Whittaker E T and Watson G N 1927 *A Course of Modern Analysis* 4th edn (Victoria: Cambridge University Press)
[29] Demiralp E and Beker H 2003 *J. Phys. A: Math. Gen.* **36** 7449–59
[30] Courant R and Hilbert D 1989 *Methods of Mathematical Physics* vol I (New York: Wiley)
[31] Landau L D and Lifshitz E M 1977 *Quantum Mechanics,* Non-Relativistics Theory 3rd revised edn (Oxford: Pergamon)
[32] Reed M and Simon B 1978 *Mehods of Modern Mathematical Physics* vol IV (San Diego: Academic)